Analytical one-factor pricing model for energy vanilla options

Boris Skorodumov*
Quantitative Associate
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One-factor model [1, 2] for forward prices was used to price vanilla options. The mean reversion rate and forward volatility were extracted from available market prices of calls and puts via calibration process. After calibration, one-factor analytical model can be used to deduce option prices with strikes, which are not available in the market.

Recently [3] we analyzed the historical data of Henry Hub, WTI Crude and Brent Crude spot prices within the scope of one-factor model of mean reverting commodity [1, 2]. One of the important characteristics of one-factor model is it’s analytical representation. Analytical pricing is important for a number of following reasons:

- It allows for fast pricing
- It aids the calibration process as each parameter in search routine can be effectively evaluated
- It is easier to derive and analyze the risk sensitivities
- Models that yield closed-form solutions, usually have fewer assumptions and parameters, which helps to understand better modeling process

In this report, we are going to price vanilla calls and puts options on futures WTI crude oil contracts within one-factor model and calibrate it to available market prices. The one-factor model [1] can be represented by the equation:

$$dS_t = \alpha S_t(\mu(t) - \ln S_t)dt + \sigma S_t dZ_t$$  \hspace{1cm} (1)

The equation 1 represents the process for single-factor spot energy price $S_t$ where $\ln S_t$ mean-reverts with speed $\alpha$ to long-term mean $\mu(t)$. It has constant volatility $\sigma$ with single source of risk $dZ_t$. Since most of the contract in energy markets are written on futures contract, the equivalent forward curve process can be derived from equation 1:

$$dF(t, T) = \frac{\partial F(0, t)}{\partial t} + \alpha (\ln F(0, t) - \ln S(t)) + \sigma^2 (1 - \exp^{-2\alpha t})$$  \hspace{1cm} (2)

$F(t, T)$ represents the price of the forward energy contract observed at time $t$ with maturity $T$. The forward curve process represented by equation 2 implies that the volatility of longer maturing contract should be less than volatility of shorter maturing contract. Also, it should be dependent on spot volatility $\sigma$. Figure 1 plots overall forward volatility extracted from 24 forward contracts obtained from a historical standard deviation of the WTI crude oil futures from November, 2007 to February, 2008, from November, 2004 to February, 2005, and from November, 1999 to February, 2000. Also is shows exponential function $f(\alpha, \sigma) = \sigma e^{-\alpha (T-t)}$ fitted to forward volatility curve. The parameters determining the level of volatility of $f(\alpha, \sigma)$ have been set to match the historically observed spot price volatility. It can be noticed that in order to match short-dated contract volatilities, the $f(\alpha, \sigma)$ forces the volatility for longer-maturity contracts to go to zero too quickly. This effect can be taken into account by multi-factor model [4] representation, which we are going to investigate in consequent reports.

The spot price process implied by equation 2 is

$$\frac{dS_t}{S_t} = \left[\frac{\partial F(0, t)}{\partial t} + \alpha (\ln F(0, t) - \ln S(t)) + \sigma^2 (1 - \exp^{-2\alpha t})\right]dt$$  \hspace{1cm} (3)

In order to have consistency between one-factor Schwartz model [1] in equation 1 and equation 3, the long term drift, $\mu(t)$, should be adjusted as

$$\mu(t) = \frac{\partial F(0, t)}{\partial t} + \ln F(0, t) + \frac{\sigma^2}{4}(1 - \exp^{-2\alpha t})$$  \hspace{1cm} (4)

Under the model described by Equations 2 and 3, the price at time $t$ of a European call option with strike price

*Electronic address: Boris.Skorodumov@mitenergy.com
matures at time \( t \) on a forward contract that matures at time \( s \), is given by:

\[
C(t, F(t, s), K, T, s) = e^{-r(T-t)} \times [F(t, s)N(h) - KN(h - \sqrt{w})]
\]

where \( h \) defined as

\[
h = \frac{ln(F(t, s)/K)}{\sqrt{w}}
\]

and \( w \) defined as

\[
w(t, T, s) = \frac{\sigma^2}{2\alpha} (e^{-2\alpha(s-T)} - e^{-2\alpha(s-t)})
\]

The formula for standard European put options can be obtained by put-call parity:

\[
P(t, F(t, s), K, T, s) = e^{-r(T-t)} \times [-F(t, s)N(-h) - KN(\sqrt{w} - h)]
\]

As an example, we show the implied estimation process for WTI crude oil futures options which is traded on the New York Mercantile Exchange (NYMEX) for September 19, 2008. Although these options have American-style exercise, we assume for this example they have European-style exercise, so that we can use Equation 5, 8 option pricing formulas.

In Table I, Market prices show the prices for WTI crude oil call and put futures options for maturities between November 2008 and January 2009, and for strikes ranging from 84 to 88.5. The maturity dates for both options and futures contracts were obtained from the WTI calendar. Model prices shows the model prices obtained via equation 5 and 8. The model prices are determined once the volatility curve is specified, which means specifying \( \alpha \) and \( \sigma \) for this model.

We employed a standard numerical search routine to find \( \alpha \) and \( \sigma \) by minimizing the squared percentage errors across all the options in Table I. In this way, the calibration is a best fit to the full set of market data. The values found in this way yield \( \alpha = 3.67 \) and \( \sigma = 75.5\% \).

Once model parameters \( \alpha \) and \( \sigma \) were extracted, it can be used to price vanilla options with different strikes and maturities.
TABLE I: Calibration of the one-factor model to WTI Crude oil futures options prices

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Model Prices

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APPENDIX A: ONE-FACTOR SPOT PRICE SDE

From Equation 2 we have that forward prices satisfy the following stochastic differential equation (SDE):

$$dF(t, T) = \sigma e^{-\alpha(T-t)} dZ_t$$

(A1)

From Ito’s lemma we have:

$$d\ln(F(t, T)) = \sigma(t, T) dZ_t - \frac{1}{2} \sigma^2(t, T) dt$$

(A2)

By integrating Equation A2 we have

$$F(t, T) = F(0, T) e^{-\frac{1}{2} \int_0^t \sigma(u) \sigma(u) du + \int_0^t \sigma(u) dZ_u}$$

(A3)

Since $S(t) = F(t, t)$ we have

$$S(t) = F(0, t) e^{-\frac{1}{2} \int_0^t \sigma^2(u) du + \int_0^t \sigma(u) dZ_u}$$

(A4)

By differentiating equation A4 we obtain

$$\frac{dS(t)}{S(t)} = \left[ \frac{\partial F(0, t)}{\partial t} + \sigma F(0, t) \frac{\partial \sigma(u)}{\partial t} du + \int_0^t \frac{\partial \sigma(u)}{\partial t} dZ(u) \right] dt + \sigma(t, t) dZ(t)$$

(A5)

In one-factor model, the volatility function is defined as

$$\sigma(t, T) = se^{-\alpha(T-t)}$$

(A6)

After integration and rearranging equation A5 we have

$$\frac{dS(t)}{S(t)} = \left[ \frac{\partial F(0, t)}{\partial t} + \alpha(\ln F(0, t) - \ln S(t)) \right] dt + \sigma(t, t) dZ(t)$$

(A7)

The equation A7 was obtained in L. Clewlow and C. Strickland [4]. In order to match Schwartz model with SDE for spot price

$$dS(t) = \alpha S(t)(\mu(t) - \ln S(t)) dt + \sigma S(t) dZ(t)$$

(A8)

the drift term needs to be equal to

$$\mu(t) = \frac{\partial F(0, t)}{\partial t} + \ln F(0, t) + \frac{\sigma^2}{4} (1 - e^{-2\alpha t})$$

(A9)